Constrained Optimal Synthesis and Robustness Analysis by Randomized Algorithms*

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Abstract

In this paper, we consider robust control using randomized algorithms. We extend the existing order statistics distribution theory to the general case in which the distribution of population is not assumed to be continuous and the order statistics is associated with certain constraints. In particular, we derive an inequality on distribution for related order statistics. Moreover, we also propose two different approaches in searching reliable solutions to the robust analysis and optimal synthesis problems under constraints. Furthermore, minimum computational effort is investigated and bounds for sample size are derived.

1 Introduction

It is now well known that many deterministic worst-case robust analysis and synthesis problems are NP hard, which means that the exact analysis and synthesis of the corresponding robust control problems may be computational demanding [5, 13]. On the other hand, the deterministic worst-case robustness measures may be quite conservative due to overbounding of the system uncertainties. As pointed out in [9] by Khargonekar and Tikku, the difficulties of deterministic worst-case robust control problems are inherent to the problem formulations and a major change of the paradigm is necessary. An alternative to the deterministic approach is the probabilistic approach which has been studied extensively by Stengel and co-workers, see for example, [10, 11], and references therein. Aimed at breaking through the NP-hardness barrier and reducing the conservativeness of the deterministic robustness measures, the probabilistic approach has recently received a renewed attention in the work

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by Barmish and Lagoa [4], Barmish, Lagoa, and Tempo [2], Barmish and Polyak [3], Khargonekar and Tikku [9], Bai, Tempo, and Fu [1], Tempo, Bai, and Dabbene [12], Yoon and Khargonekar [14], Zhu, Huang and Doyle [16], Chen and Zhou [6, 7] and references therein.

In addition to its low computational complexity, the advantages of randomized algorithms can be found in the flexibility and adaptiveness in dealing with control analysis or synthesis problems with complicated constraints or in the situation of handling nonlinearities. The robust control analysis and synthesis problems under constraints are, in general, very hard to deal with in the deterministic framework. For example, it is well-known that a multi-objective control problem involving mixed H_2 and H_∞ objectives are very hard to solve even though there are elegant solutions to the pure H_2 or H_∞ problems [15].

In this paper, we first show that most of the robust control problems can be formulated as constrained optimal synthesis or robust analysis problems. Since the exact robust analysis or synthesis is, in general, impossible, we seek a 'reliable' solution by using randomized algorithms. Roughly speaking, by 'reliability' we mean how the solution resulted by randomized algorithms approaches the exact one. In this paper, we measure the degree of 'reliability' in terms of accuracy $1-\varepsilon$ and confidence level $1-\delta$. Actually, terminologies like 'accuracy' and 'confidence level' have been used in [12] and [9] where accuracy $1-\varepsilon$ is referred as an upper bound of the absolute *volume* of a subset of parameter space $\mathbf{Q}$. However, in this paper, we emphasis that the accuracy $1-\varepsilon$ is an upper bound for the *ratio* of *volume* of the *constrained subset* $\mathbf{Q_C} := \{$ constraint set $\mathbf{C}$ holds, $q \in \mathbf{Q}\}$ with respect to the volume of parameter space $\mathbf{Q}$. For example, when estimating the minimum of a quantity $u(q)$ over $\mathbf{Q_C}$, the ratio may be $\frac{\text{volume of } \{u(q)\geq \hat{u}_{min},\ q\in \mathbf{Q_C}\}}{\text{volume of } \mathbf{Q_C}}$ where $\hat{u}_{min}$ is an estimate resulted by randomized algorithms for quantity $u(q)$. We can see that the ratio of volume is a better indicator of the 'reliability' than the absolute volume of $\{u(q) \geq \hat{u}_{min},\ q \in \mathbf{Q_C}\}$.

Based on this measure of 'reliability', we propose two different approaches aimed at seeking a solution to the robust analysis or optimal synthesis problem with a certain a priori specified degree of 'reliability'. One is the **direct approach**. The key issue is to determine the number of samples needed to be generated from the parameter space $\mathbf{Q}$ for a given reliability measure. Actually, Khargonekar and Tikku in [9] have applied similar approach to stability margin problem, though the measure of 'reliability' is in terms of the absolute volume. In that paper, a sufficient condition is derived on the sample size required to come up with a 'reliable' estimate of the robust stability margin (See Theorem 3.3 in [9]). In this paper, we also derive the bound of sample size and give the sufficient and necessary condition for the existence of minimum distribution-free samples size. Our result shows that, the bound of sample size necessarily involves $\rho :=$ volume of $\mathbf{Q_C}$. Thus estimating ρ becomes essential. Unfortunately, estimating ρ is time-consuming and the resulted sample size is not accurate. To overcome this difficulty, we propose and strongly advocate another approach—the **indirect approach**. The key issue is to determine the *constrained sample size*, which is the number

of samples needed that fall into the constrained subset $\mathbf{Q_C}$. We derive bounds of constrained sample size and give the sufficient and necessary condition for the existence of minimum distribution-free constrained samples size. The bounds do not involve ρ and can be computed exactly. This result makes it possible to obtain a reliable solution without estimating the volume of the constrained parameter subset $\mathbf{Q_C}$.

This paper is organized as follows. Section 2 presents the problem formulation and motivations. In Section 3, we derive the exact distribution of related order statistics without the continuity assumption. Distribution free tolerance interval and estimation of quantity range is discussed in Section 4. Section 5 gives the minimum sample size under various assumptions.

2 Preliminary and Problem Formulation

Let $q = [q_1 \ \cdots \ q_n]^T$ be a vector of a control system's parameters, bounded in a compact set $\mathbf{Q}$, i.e., $q \in \mathbf{Q}$. Let $\mathbf{C}$ be a set of constraints that q must satisfy. Define the *constrained subset* of $\mathbf{Q}$ by $\mathbf{Q_C} := \{\ \mathbf{C} \text{ holds},\ q \in \mathbf{Q}\}$. Let $u(q)$ denote a performance index function. In many applications, we are concerned with a performance index function $u(q)$ of the system under the set of constraints $\mathbf{C}$. It is natural to ask the following questions:

- What is $\min_{\mathbf{Q_C}} u(q)$ (or $\max_{\mathbf{Q_C}} u(q)$)?
- What is the value of q at which $u(q)$ achieves $\min_{\mathbf{Q_C}} u(q)$ (or $\max_{\mathbf{Q_C}} u(q)$)?

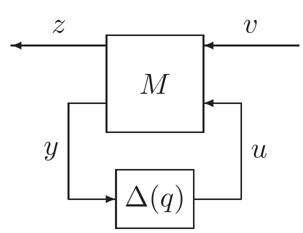

Figure 1: Uncertain System

Consider, for example, an uncertain system shown in Figure 1. Denote the transfer function from v to z by T_{zv} and suppose that T_{zv} has the following state space realization $T_{zv} = \left[\begin{array}{c|c} A(q) & B(q) \\ \hline C(q) & D(q) \end{array}\right]$. We can now consider several robustness problems:

- Robust stability: Let $\mathbf{Q_C} = \mathbf{Q}$ and $u(q) := \max_i \operatorname{Re} \lambda_i(A(q))$ where $\lambda_i(A)$ denotes the i-th eigenvalue of A. Then the system is robustly stable if $\max_{q \in \mathbf{Q_C}} \ u(q) < 0$.
- Stability margin: Assume that $\Delta(q)$ belongs to the class of allowable perturbations $\mathbf{\Delta}$ which has a certain block structure. For a given real number γ, let $\mathbf{\Delta}_\gamma$ denote the subset of perturbations

in $\mathbf{\Delta}$ with size at most γ, i.e., $\mathbf{\Delta}_\gamma := \{\Delta(q) \in \mathbf{\Delta} \ : \ \bar{\sigma}(\Delta(q)) \leq \gamma\}$. The *robustness measure* γ_{opt} is defined as the smallest allowable perturbation that destabilizes the feedback interconnection. Let γ_0 be an upper bound for γ_{opt}. Define parameter space $\mathbf{Q}$ by $\mathbf{Q} := \{q \ : \ \Delta(q) \in \mathbf{\Delta}_{\gamma_0}\}$ and constrained subset $\mathbf{Q_C}$ by $\mathbf{Q_C} := \{q : q \in \mathbf{Q}$ and $A(q)$ is unstable$\}$. Let $u(q) := \bar{\sigma}(\Delta(q))$. It follows that the stability margin problem is equivalent to computing $\gamma_{opt} = \min_{\mathbf{Q_C}} u(q)$.

- Robust performance: Suppose $A(q)$ is stable for all $q \in \mathbf{Q}$. Define $u(q) := ||T_{zv}||_\infty$. Then the robust performance problem is to determine if $\max_{q\in\mathbf{Q}} u(q) \leq \gamma$ is satisfied for some prespecified $\gamma > 0$.

- Performance range: Let $\mathbf{Q_C} \subseteq \mathbf{Q}$ be a given set of parameters such that $A(q)$ is stable for all $q \in \mathbf{Q_C}$. Define again $u(q) := ||T_{zv}||_\infty$. Then the problem of determining the range of the system's H_∞ performance level can be formulated as finding $\min_{q\in\mathbf{Q_C}} u(q)$ and $\max_{q\in\mathbf{Q_C}} u(q)$.

As another example, consider the problem of designing a controller $K(q)$ for an uncertain system $P(s)$. Suppose that q is a vector of controller parameters to be designed and that the controller is connected with $P(s)$ in a lower LFT setup. Let the transfer function of the whole system be denoted as $F_l(P(s), K(q))$. Suppose that $F_l(P(s), K(q))$ has the following state space realization $F_l(P(s), K(q)) = \left[\begin{array}{c|c} A_s(q) & B_s(q) \\ \hline C_s(q) & D_s(q) \end{array}\right]$. Then we can formulate the problem as a constrained optimal synthesis problem by defining a performance index $u(q) := ||F_l(P(s), K(q))||_\infty$ and restricting parameter q to $\mathbf{Q_C} := \{\ \max_i \operatorname{Re} \lambda_i(A_s(q)) < -\alpha, \ q \in \mathbf{Q}\}$ where $\alpha > 0$ is not too small for a stability margin. Then the H_∞ design problem is to determine a vector of parameters achieving $\min_{q\in\mathbf{Q_C}} u(q)$.

2.1 A Measure of Reliability

Since the exact solution to the analysis or synthesis problem is impossible. Measuring how the solution resulted by the randomized algorithm approaches the exact one becomes essential. We shall first introduce the concept of *volume* proposed in [9]. Let $w(q)$ denote the cumulative distribution function of q. For a subset $\mathbf{U} \subseteq \mathbf{Q}$, the volume of $\mathbf{U}$, denoted by $vol_w\{\mathbf{U}\}$, is defined by $vol_w\{\mathbf{U}\} := \int_{q\in\mathbf{U}} dw(q)$. Define $\rho := \frac{vol_w\{\mathbf{Q_C}\}}{vol_w\{\mathbf{Q}\}}$. Then it follows that $\rho = vol_w\{\mathbf{Q_C}\}$ since $vol_w(\mathbf{Q}) = 1$. We assume throughout this paper that $u(q)$ is a measurable function of q and that $vol_w\{\mathbf{Q_C}\} > 0$. We also assume throughout this paper that $\varepsilon, \delta \in (0, 1)$. Let $\hat{u}_{min}$ and $\hat{u}_{max}$ be the estimates of $\min_{q\in\mathbf{Q_C}} u(q)$ and $\max_{q\in\mathbf{Q_C}} u(q)$ respectively. Note that $\hat{u}_{max}$ and $\hat{u}_{min}$ are random variables resulted by randomized algorithms. A reliable estimate of $\hat{u}_{min}$ should guarantee $\Pr\left\{\frac{vol_w\{u(q)\geq\hat{u}_{min},\ q\in\mathbf{Q_C}\}}{vol_w\{\mathbf{Q_C}\}} \geq 1-\varepsilon\right\} \geq 1-\delta$ for a small ε and a small δ. Similarly, a reliable estimate of $\hat{u}_{max}$ should guarantee $\Pr\left\{\frac{vol_w\{u(q)\leq\hat{u}_{max},\ q\in\mathbf{Q_C}\}}{vol_w\{\mathbf{Q_C}\}} \geq 1-\varepsilon\right\} \geq 1-\delta$ for a small ε and a small δ.

2.2 Two Different Approaches

- **Indirect Approach** Generate i.i.d. samples q^i for q by the same distribution function $w(q)$. Continue the sampling process until we obtain N_c observations of q which belong to $\mathbf{Q_C}$. Let L be the number of i.i.d. experiments when this sampling process is terminated. Then L is a random number with distribution satisfying $\sum_{l=N_c}^{\infty} \Pr\{L = l\} = 1$ and we can show that $E[L] = \frac{N_c}{\rho}$. Let the observations which belong to $\mathbf{Q_C}$ be denoted as $q_c^i, \;\; i = 1, \cdots, N_c$. Define order statistics $\hat{u}_i, \; i = 1, \cdots, N_c$ as the ith smallest one of the set of observations $\{u(q_c^i) : \;\; i = 1, \cdots, N_c\}$, i.e., $\hat{u}_1 \leq \cdots \leq \hat{u}_{N_c}$. Obviously, it is reasonable to take $\hat{u}_1$ as an estimate for $\min_{\mathbf{Q_C}} u(q)$ and $\hat{u}_{N_c}$ as an estimate for $\max_{\mathbf{Q_C}} u(q)$ if N_c is sufficiently large. Henceforth, we need to know N_c which guarantees $\Pr\left\{\frac{vol_w\{u(q) \geq \hat{u}_1, \, q \in \mathbf{Q_C}\}}{vol_w\{\mathbf{Q_C}\}} \geq 1 - \varepsilon\right\} \geq 1 - \delta$ and (or) $\Pr\left\{\frac{vol_w\{u(q) \leq \hat{u}_{N_c}, \, q \in \mathbf{Q_C}\}}{vol_w\{\mathbf{Q_C}\}} \geq 1 - \varepsilon\right\} \geq 1 - \delta$. We call N_c *constrained sample size.*

- **Direct Approach** Let $q^1, \cdots, q^N$ be N i.i.d samples generated by the same distribution function $w(q)$. Define $\mathbf{S} := \left\{q^1, \cdots, q^N\right\} \bigcap \mathbf{Q_C}$. Let M be the number of the elements in $\mathbf{S}$. Then M is a random number. If $M \geq 1$ we denote the elements of $\mathbf{S}$ as $q_c^i, \;\; i = 1, \cdots, M$. Define order statistics $\hat{u}_i, \; i = 1, \cdots, M$ as the ith smallest one of the set of observations $\{u(q_c^i) : \;\; i = 1, \cdots, M\}$, i.e., $\hat{u}_1 \leq \cdots \leq \hat{u}_M$. In particular, let $\hat{u}_{min} = \hat{u}_1$ and $\hat{u}_{max} = \hat{u}_M$. We need to know N which guarantees $\Pr\left\{\frac{vol_w\{u(q) \geq \hat{u}_{min}, \, q \in \mathbf{Q_C}\}}{vol_w\{\mathbf{Q_C}\}} \geq 1 - \varepsilon\right\} \geq 1 - \delta$ and (or) $\Pr\left\{\frac{vol_w\{u(q) \leq \hat{u}_{max}, \, q \in \mathbf{Q_C}\}}{vol_w\{\mathbf{Q_C}\}} \geq 1 - \varepsilon\right\} \geq 1 - \delta$. We call N *global sample size.*

3 Exact Distribution

Define $F_u(\gamma) := \frac{vol_w\{u(q) \leq \gamma, \, q \in \mathbf{Q_C}\}}{vol_w\{\mathbf{Q_C}\}}$. To compute the probabilities involved in Section 2, it is important to know the associated distribution of any k random variables $F_u(\hat{u}_{i_1}), \; \cdots, \; F_u(\hat{u}_{i_k}), \;\; 1 \leq i_1 < \cdots < i_k \leq N_c, \;\; 1 \leq k \leq N_c$ where $\hat{u}_{i_s}, \; s = 1, \cdots, k$ is order statistics in the context of the indirect approach. First, we shall established the following lemma.

Lemma 1 *Let U be a random variable with uniform distribution over $[0, 1]$ and $\hat{U}_n, \;\; n = 1, \cdots, N$ be the order statistics of U, i.e., $\hat{U}_1 \leq \cdots \leq \hat{U}_N$. Let $0 = t_0 < t_1 \cdots < t_k \leq 1$. Define*

$$G_{j_1, \cdots, j_k}(t_1, \cdots, t_k) := (1 - t_k)^{N - \sum_{l=1}^{k} j_l} \prod_{s=1}^{k} \binom{N - \sum_{l=1}^{s-1} j_l}{j_s} (t_s - t_{s-1})^{j_s}$$

and

$$\mathbf{I}_{i_1, \cdots, i_k} := \left\{(j_1, \cdots, j_k) : \;\; i_s \leq \sum_{l=1}^{s} j_l \leq N, \;\; s = 1, \cdots, k\right\}.$$

Then $\Pr\left\{\hat{U}_{i_1} \leq t_1, \cdots, \hat{U}_{i_k} \leq t_k\right\} = \sum_{(j_1, \cdots, j_k) \in \mathbf{I}_{i_1, \cdots, i_k}} G_{j_1, \cdots, j_k}(t_1, \cdots, t_k)$.

Proof. Let j_s be the number of samples of U which fall into $(t_{s-1}, t_s]$, $s = 1, \cdots, k$. Then the number of samples of U which fall into $[0, t_s]$ is $\sum_{l=1}^{s} j_l$. It is easy to see that the event $\left\{\hat{U}_{i_s} \leq t_s\right\}$ is equivalent to event $\{i_s \leq \sum_{l=1}^{s} j_l \leq N\}$. Furthermore, the event $\left\{\hat{U}_{i_1} \leq t_1, \cdots, \hat{U}_{i_k} \leq t_k\right\}$ is equivalent to the event $\{i_s \leq \sum_{l=1}^{s} j_l \leq N, \ \ s = 1, \cdots, k\}$. Therefore,

$$\Pr\left\{\hat{U}_{i_1} \leq t_1, \cdots, \hat{U}_{i_k} \leq t_k\right\} = \sum_{(j_1,\cdots,j_k)\in \mathbf{I}_{i_1,\cdots,i_k}} \prod_{s=1}^{k} \binom{N - \sum_{l=1}^{s-1} j_l}{j_s} (t_s - t_{s-1})^{j_s} \ (1-t_k)^{N-\sum_{l=1}^{k} j_l}$$
$$= \sum_{(j_1,\cdots,j_k)\in \mathbf{I}_{i_1,\cdots,i_k}} G_{j_1,\cdots,j_k}(t_1, \cdots, t_k).$$

□

Theorem 1 *Let* $0 = t_0 < t_1 \leq \cdots \leq t_k \leq 1$ *and* $x_0 = 0, \ x_{k+1} = 1, \ i_0 = 0, \ i_{k+1} = N+1$. *Define* $f_{i_1,\cdots,i_k}(x_1, \cdots, x_k) := \prod_{s=0}^{s=k} N! \ \frac{(x_{s+1}-x_s)^{i_{s+1}-i_s-1}}{(i_{s+1}-i_s-1)!}$ *and* $\mathbf{D}_{p_1,\cdots,p_k} := \{(x_1, \cdots, x_k) : \ \ 0 \leq x_1 \leq \cdots \leq x_k, \ x_s \leq p_s, \ s = 1, \cdots, k\}$. *Define* $F(t_1, \cdots, t_k) := \Pr\{F_u(\hat{u}_{i_1}) < t_1, \cdots, F_u(\hat{u}_{i_k}) < t_k\}$ *and* $\tau_s := \sup_{\{x: \ F_u(x)<t_s\}} F_u(x), \ \ s = 1, \cdots, k$. *Then*

$$F(t_1, \cdots, t_k) = \int_{\mathbf{D}_{\tau_1,\cdots,\tau_k}} f_{i_1,\cdots,i_k}(x_1, \cdots, x_k) \ dx_1 \cdots dx_k \leq \int_{\mathbf{D}_{t_1,\cdots,t_k}} f_{i_1,\cdots,i_k}(x_1, \cdots, x_k) \ dx_1 \cdots dx_k$$

and the last equality holds if and only if $\exists x_s^*$ such that $\Pr\{u(q) < x_s^* \mid q \in \mathbf{Q_C}\} = t_s, \ \ s = 1, \cdots, k$.

Proof. Define $\alpha_0 := -\infty$ and $\alpha_s := \sup\{x : \ F_u(x) < t_s\}, \ \ \alpha_s^- := \alpha_s - \epsilon, \ \ s = 1, \cdots, k$ where $\epsilon > 0$ can be arbitrary small. Let $\phi_s := F_u(\alpha_s^-), \ \ s = 1, \cdots, k$. We can show that $\phi_l < \phi_s$ if $\alpha_l < \alpha_s, \ 1 \leq l < s \leq k$. In fact, if this is not true, we have $\phi_l = \phi_s$. Because ϵ can be arbitrarily small, we have $\alpha_s^- \in (\alpha_l, \ \alpha_s)$. Notice that $\alpha_l = \min\{x : \ F_u(x) \geq t_l\}$, we have $t_l \leq \phi_s = \phi_l$. On the other hand, by definition we know that $\alpha_l^- \in \{x : \ F_u(x) < t_l\}$ and thus $\phi_l = F_u(\alpha_l^-) < t_l$, which is a contradiction. Notice that $F_u(\gamma)$ is nondecreasing and right-continuous, we have $\alpha_1 \leq \cdots \leq \alpha_k$ and $0 \leq \phi_1 \leq \cdots \leq \phi_k \leq 1$ and that event $\{F_u(\hat{u}_{i_s}) < t_s | \ L = l\}$ is equivalent to the event $\{\hat{u}_{i_s} < \alpha_s | \ L = l\}$. Furthermore, event $\{F_u(\hat{u}_{i_1}) < t_1, \ \cdots, \ F_u(\hat{u}_{i_k}) < t_k | \ L = l\}$ is equivalent to event $\{\hat{u}_{i_1} < \alpha_1, \ \cdots, \ \hat{u}_{i_k} < \alpha_k | \ L = l\}$ which is defined by k constraints $\hat{u}_{i_s} < \alpha_s, \ s = 1, \cdots, k$. For every $l < k$, delete constraint $\hat{u}_{i_l} < \alpha_l$ if there exists $s > l$ such that $\alpha_s = \alpha_l$. Let the remaining constraints be $\hat{u}_{i_s'} < \alpha_s', \ s = 1, \cdots, k'$ where $\alpha_1' < \cdots < \alpha_{k'}'$. Since all constraints deleted are actually redundant, it follows that event $\{\hat{u}_{i_1} < \alpha_1, \ \cdots, \ \hat{u}_{i_k} < \alpha_k | \ L = l\}$ is equivalent to event $\left\{\hat{u}_{i_1'} < \alpha_1', \ \cdots, \ \hat{u}_{i_{k'}'} < \alpha_{k'}' | \ L = l\right\}$. Now let j_s be the number of observations $u(q_c^i)$ which fall into $[\alpha_{s-1}', \alpha_s'), \ \ s = 1, \cdots, k'$. Then the number of observations $u(q_c^i)$ which fall into $(-\infty, \alpha_s')$ is $\sum_{l=1}^{s} j_l$. It is easy to see that the event $\left\{\hat{u}_{i_s'} < \alpha_s' | \ L = l\right\}$ is equivalent to the event

$\left\{ i_s^{'} \leq \sum_{l=1}^{s} j_l \leq N \right\}$. Furthermore, the event $\left\{ \hat{u}_{i_1^{'}} < \alpha_1^{'}, \ \cdots, \ \hat{u}_{i_{k^{'}}^{'}} < \alpha_{k^{'}}^{'} | \ L = l \right\}$ is equivalent to event $\left\{ i_s^{'} \leq \sum_{l=1}^{s} j_l \leq N, \ \ s = 1, \cdots, k^{'} \right\}$. Therefore

$$
\begin{aligned}
& \Pr \left\{ F_u(\hat{u}_{i_1}) < t_1, \ \cdots, \ F_u(\hat{u}_{i_k}) < t_k | \ L = l \right\} \\
= & \Pr \left\{ \hat{u}_{i_1} < \alpha_1, \ \cdots, \ \hat{u}_{i_k} < \alpha_k | \ L = l \right\} = \Pr \left\{ \hat{u}_{i_1^{'}} < \alpha_1^{'}, \ \cdots, \ \hat{u}_{i_{k^{'}}^{'}} < \alpha_{k^{'}}^{'} | \ L = l \right\} \\
= & \sum_{(j_1, \cdots, j_{k^{'}}) \in \mathbf{I}_{i_1^{'}, \cdots, i_{k^{'}}^{'}}} \prod_{s=1}^{k^{'}} \binom{N - \sum_{l=1}^{s-1} j_l}{j_s} \left[F_u(\alpha_s^{'-}) - F_u(\alpha_{s-1}^{'-}) \right]^{j_s} \left[1 - F_u(\alpha_{k^{'}}^{'-}) \right]^{N - \sum_{l=1}^{k^{'}} j_l} \\
= & \sum_{(j_1, \cdots, j_{k^{'}}) \in \mathbf{I}_{i_1^{'}, \cdots, i_{k^{'}}^{'}}} G_{j_1, \cdots, j_{k^{'}}} \left(\phi_1^{'}, \cdots, \phi_{k^{'}}^{'} \right).
\end{aligned}
$$

Now consider event $\left\{ \hat{U}_{i_1} \leq \phi_1, \ \cdots, \ \hat{U}_{i_k} \leq \phi_k \right\}$. For every $l < k$, delete constraint $\hat{U}_{i_l} \leq \phi_l$ if there exists $s > l$ such that $\phi_s = \phi_l$. Notice that $\phi_s = F_u(\alpha_s^-)$ and $\phi_l < \phi_s$ if $\alpha_l < \alpha_s, \ 1 \leq l < s \leq k$, the remaining constraints must be $\hat{U}_{i_s^{'}} \leq \phi_s^{'}, \ s = 1, \cdots, k^{'}$ where $\phi_s^{'} = F_u(\alpha_s^{'-}), \ s = 1, \cdots, k^{'}$ and $\phi_1^{'} < \cdots < \phi_{k^{'}}^{'}$. Since all constraints deleted are actually redundant, it follows that event $\left\{ \hat{U}_{i_1} \leq \phi_1, \ \cdots, \ \hat{U}_{i_k} \leq \phi_k \right\}$ is equivalent to event $\left\{ \hat{U}_{i_1^{'}} \leq \phi_1^{'}, \ \cdots, \ \hat{U}_{i_{k^{'}}^{'}} \leq \phi_{k^{'}}^{'} \right\}$. By Theorem 2.2.3 in [8] and Lemma 1

$$
\begin{aligned}
& \int_{\mathbf{D}_{\phi_1, \cdots, \phi_k}} f_{i_1, \cdots, i_k}(x_1, \cdots, x_k) \, dx_1 \cdots dx_k = \Pr \left\{ \hat{U}_{i_1} \leq \phi_1, \ \cdots, \ \hat{U}_{i_k} \leq \phi_k \right\} \\
= & \Pr \left\{ \hat{U}_{i_1^{'}} \leq \phi_1^{'}, \ \cdots, \ \hat{U}_{i_{k^{'}}^{'}} \leq \phi_{k^{'}}^{'} \right\} = \sum_{(j_1, \cdots, j_{k^{'}}) \in \mathbf{I}_{i_1^{'}, \cdots, i_{k^{'}}^{'}}} G_{j_1, \cdots, j_{k^{'}}} \left(\phi_1^{'}, \cdots, \phi_{k^{'}}^{'} \right).
\end{aligned}
$$

Therefore, $\Pr \left\{ F_u(\hat{u}_{i_1}) < t_1, \ \cdots, \ F_u(\hat{u}_{i_k}) < t_k | \ L = l \right\} = \int_{\mathbf{D}_{\phi_1, \cdots, \phi_k}} f_{i_1, \cdots, i_k}(x_1, \cdots, x_k) \ dx_1 \cdots dx_k$. It follows that

$$
\begin{aligned}
& F(t_1, \cdots, t_k) = \Pr \left\{ F_u(\hat{u}_{i_1}) < t_1, \ \cdots, \ F_u(\hat{u}_{i_k}) < t_k \right\} \\
= & \sum_{l=N_c}^{\infty} \Pr \left\{ F_u(\hat{u}_{i_1}) < t_1, \ \cdots, \ F_u(\hat{u}_{i_k}) < t_k \mid L = l \right\} \Pr \left\{ L = l \right\} \\
= & \sum_{l=N_c}^{\infty} \int_{\mathbf{D}_{\phi_1, \cdots, \phi_k}} f_{i_1, \cdots, i_k}(x_1, \cdots, x_k) \, dx_1 \cdots dx_k \Pr \left\{ L = l \right\}
\end{aligned}
$$

Notice that $\sum_{l=N_c}^{\infty} \Pr \left\{ L = l \right\} = 1$. We have

$$
\begin{aligned}
& F(t_1, \cdots, t_k) = \int_{\mathbf{D}_{\phi_1, \cdots, \phi_k}} f_{i_1, \cdots, i_k}(x_1, \cdots, x_k) \, dx_1 \cdots dx_k \sum_{l=N_c}^{\infty} \Pr \left\{ L = l \right\} \\
= & \int_{\mathbf{D}_{\phi_1, \cdots, \phi_k}} f_{i_1, \cdots, i_k}(x_1, \cdots, x_k) \, dx_1 \cdots dx_k.
\end{aligned}
$$

By the definitions of τ_s and ϕ_s, we know that $\mathbf{D}_{\tau_1,\cdots,\tau_k}$ is the closure of $\mathbf{D}_{\phi_1,\phi_2,\cdots,\phi_k}$, i.e., $\mathbf{D}_{\tau_1,\cdots,\tau_k} = \bar{\mathbf{D}}_{\phi_1,\phi_2,\cdots,\phi_k}$ and that their Lebesgue measures are equal. It follows that

$$F(t_1, \cdots, t_k) = \int_{\mathbf{D}_{\tau_1,\cdots,\tau_k}} f_{i_1,\cdots,i_k}(x_1, \cdots, x_k)\, dx_1 \cdots dx_k.$$

Notice that $\tau_s \leq t_s, \;\; s = 1, \cdots, k$, we have $\mathbf{D}_{\tau_1,\cdots,\tau_k} \subseteq \mathbf{D}_{t_1,\cdots,t_k}$ and hence

$$F(t_1, \cdots, t_k) \leq \int_{\mathbf{D}_{t_1,t_2,\cdots,t_k}} f_{i_1,\cdots,i_k}(x_1, \cdots, x_k)\, dx_1 \cdots dx_k,$$

where the equality holds if and only if $\tau_s = t_s, \;\; s = 1, \cdots, k$, i.e., $\exists x_s^*$ such that $\Pr\{u(q) < x_s^* \mid q \in \mathbf{Q_C}\} = t_s, \;\; s = 1, \cdots, k.$ $\square$

Remark 1 *For the special case of $\mathbf{Q_C} = \mathbf{Q}$ and that $F_u(.)$ is absolutely continuous, $F(t_1, \cdots, t_k)$ can be obtained by combining Probability Integral Transformation Theorem and Theorem* 2.2.3 *in [8]. However, in robust control problem, the continuity of $F_u(.)$ is not necessarily guaranteed. For example, $F_u(.)$ is not continuous when uncertain quantity $u(q)$ equals to a constant in an open set of $\mathbf{Q_C}$. We can come up with many uncertain systems in which the continuity assumption for the distribution of quantity $u(q)$ is not guaranteed. Since it is reasonable to assume that $u(q)$ is measurable, Theorem 1 can be applied in general to tackle these problems without continuity assumption by a probabilistic approach. In addition, Theorem 1 can be applied to investigate the minimum computational effort to come up with a solution with a certain degree of 'reliability' for robust analysis or optimal synthesis problems under constraints.*

From the proof of Theorem 1, we can see that $F(t_1, \cdots, t_k)$ is not related to the knowledge of L, thus we have the following corollary.

Corollary 1 *Let $N_2 \geq N_1 \geq N_c$. Then* $\Pr\{F_u(\hat{u}_{i_1}) < t_1, \; \cdots, \; F_u(\hat{u}_{i_k}) < t_k \mid N_1 \leq L \leq N_2\} = F(t_1, \cdots, t_k)$.

4 Quantity Range and Distribution-Free Tolerance Intervals

In robust analysis or synthesis, it is desirable to know function $F_u(.)$ because it is actually the distribution function of quantity $u(q)$ for $q \in \mathbf{Q_C}$. However, the exact computation of function $F_u(.)$ is in general impossible. We shall extract as much as possible the information of $F_u(.)$ from observations $u(q_c^i), \; i = 1, \cdots, N_c$. Let $\mathcal{V}(N_c, \; i, \; \varepsilon) := \int_\varepsilon^1 \frac{N_c!}{(i-1)!(N_c-i)!} x^{i-1}(1-x)^{N_c-i} dx$ for $1 \leq i \leq N_c$.

Theorem 2 $\Pr\left\{\frac{vol_w\{u(q) \geq \hat{u}_m, \; q \in \mathbf{Q_C}\}}{vol_w\{\mathbf{Q_C}\}} \geq 1 - \varepsilon\right\} \geq 1 - \mathcal{V}(N_c, \; m, \; \varepsilon)$ *with the equality holds if and only if $\exists x^*$ such that $F_u(x^*) = \varepsilon$. Moreover,* $\Pr\left\{\frac{vol_w\{u(q) \leq \hat{u}_m, \; q \in \mathbf{Q_C}\}}{vol_w\{\mathbf{Q_C}\}} \geq 1 - \varepsilon\right\} \geq 1 - \mathcal{V}(N_c, \; N_c + 1 - m, \; \varepsilon)$ *with the equality holds if and only if $\exists x^*$ such that* $\Pr\{u(q) < x^* \mid q \in \mathbf{Q_C}\} = 1 - \varepsilon$.

Proof. Let $v(q) = -u(q)$. Let the cumulative distribution function of $v(q)$ be $F_v(.)$ and define order statistics $\hat{v}_i,\ \ i = 1, \cdots, N_c$ as the i-th smallest one of the set of observations $\{v(q_c^i)|\ i = 1, \cdots, N_c\}$, i.e., $\hat{v}_1 \leq \cdots \leq \hat{v}_{N_c}$. Obviously, $\hat{u}_m = -\hat{v}_{N_c+1-m}$ for any $1 \leq m \leq N_c$. It is also clear that $F_v(-x) = 1 - F_u(x^-)$, which leads to $\sup_{\{x:F_v(x)<1-\varepsilon\}} F_v(x) = 1 - \varepsilon \Longleftrightarrow \inf_{\{x:F_u(x)>\varepsilon\}} F_u(x) = \varepsilon$. Apply Theorem 1 to the case of $k = 1,\ i_1 = N_c + 1 - m$, we have

$$\Pr\{F_v(\hat{v}_{N_c+1-m}) < 1 - \varepsilon\} = \int_0^\tau \frac{N_c!}{(N_c - m)!(m-1)!} x^{N_c - m}(1-x)^{m-1} dx$$
$$\leq \int_0^{1-\varepsilon} \frac{N_c!}{(N_c - m)!(m-1)!} x^{N_c - m}(1-x)^{m-1} dx = \mathcal{V}(N_c,\ m,\ \varepsilon)$$

where $\tau = \sup\limits_{\{x:F_v(x)<1-\varepsilon\}} F_v(x)$. Therefore,

$$\Pr\left\{\frac{vol_w\{v(q) \leq \hat{v}_{N_c+1-m},\ q \in \mathbf{Q_C}\}}{vol_w\{\mathbf{Q_C}\}} \geq 1 - \varepsilon\right\} = \Pr\{F_v(\hat{v}_{N_c+1-m}) \geq 1 - \varepsilon\}$$
$$= 1 - \Pr\{F_v(\hat{v}_{N_c+1-m}) < 1 - \varepsilon\} \geq 1 - \mathcal{V}(N_c,\ m,\ \varepsilon).$$

The equality holds if and only if $\exists x^*$ such that $F_u(x^*) = \varepsilon$ because such a x^* exists if and only if $\tau = 1 - \varepsilon$. It follows that

$$\Pr\left\{\frac{vol_w\{u(q) \geq \hat{u}_m,\ q \in \mathbf{Q_C}\}}{vol_w\{\mathbf{Q_C}\}} \geq 1 - \varepsilon\right\}$$
$$= \Pr\left\{\frac{vol_w\{v(q) \leq \hat{v}_{N_c+1-m},\ q \in \mathbf{Q_C}\}}{vol_w\{\mathbf{Q_C}\}} \geq 1 - \varepsilon\right\} \geq 1 - \mathcal{V}(N_c,\ m,\ \varepsilon)$$

with the equality holds if and only if $\exists x^*$ such that $F_u(x^*) = \varepsilon$.

The second part follows by applying Theorem 1 to the case of $k = 1,\ i_1 = m$. □

It is important to note that the two conditions in Theorem 2 are much weaker than the continuity assumption which requires that for any $p \in (0, 1)$ there exists $x^\star$ such that $F_u(x^\star) = p$. The difference is visualized in Figure 2.

In general, it is important to know the probability of a quantity falling between two arbitrary samples. To that end, we have

Corollary 2 *Let $1 \leq m < n \leq N_c$. Suppose $u(q) \neq constant$ in any open set of $\mathbf{Q_C}$. Then* $\Pr\{\frac{vol_w\{\hat{u}_m < u(q) \leq \hat{u}_n,\ q \in \mathbf{Q_C}\}}{vol_w\{\mathbf{Q_C}\}} \geq 1 - \varepsilon\} = 1 - \mathcal{V}(N_c,\ N_c + 1 - n + m,\ \varepsilon)$.

Since the condition that $u(q) \neq$ *constant* in any open set of $\mathbf{Q_C}$ is equivalent to the absolute continuity assumption of $F_u(x)$ (see the proof of Theorem 3.3 in [9]), the proof of Corollary 2 can be completed by applying Theorem 1 to the case of $k = 2$, $i_1 = m$, $i_2 = n$ and $F_u(x)$ is continuous.

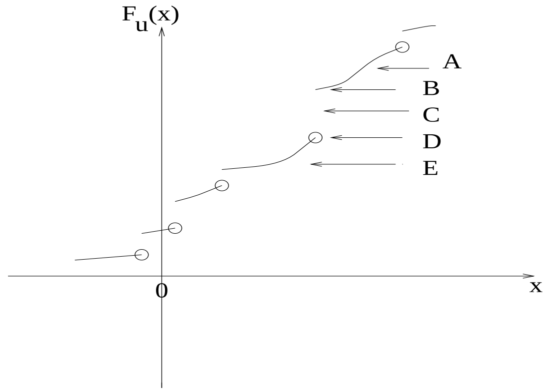

Figure 2: Cases A, B and E guarantee $\exists x^*$ such that $F_u(x^*) = \varepsilon$. Cases A, D and E guarantee $\exists x^*$ such that $\Pr\{u(q) < x^* \mid q \in \mathbf{Q_C}\} = 1 - \varepsilon$. Both conditions are violated in Case C.(The various magnitude of ε and $1-\varepsilon$ is indicated by arrows at different heights.)

5 Sample Size

The important issue of the randomized algorithms to robust analysis or optimal synthesis is to determine the minimum computational effort required to come up with a solution with a certain degree of 'reliability'. First, we consider this issue for the indirect approach.

5.1 Constrained Sample Size

To estimate $\max_{\mathbf{Q_C}} u(q)$ (or determine parameter q achieving $\max_{\mathbf{Q_C}} u(q)$), we have

Corollary 3 *Suppose that* $\exists x^*$ *such that* $\Pr\{u(q) < x^* \mid q \in \mathbf{Q_C}\} = 1 - \varepsilon$. *Then*

$$\Pr\{\frac{vol_w\{u(q) \le \hat{u}_{N_c},\ q \in \mathbf{Q_C}\}}{vol_w\{\mathbf{Q_C}\}} \ge 1 - \varepsilon\} \ge 1 - \delta$$

if and only if $N_c \ge \frac{\ln \frac{1}{\delta}}{\ln \frac{1}{1-\varepsilon}}$.

It should be noted that the results in Khargonekar and Tikku [9] and Tempo, Bai, and Dabbene [12] correspond to the sufficient part of the above Corollary for the special case of $\mathbf{Q_C} = \mathbf{Q}$.

To estimate $\min_{\mathbf{Q_C}} u(q)$ (or determine parameter q achieving $\min_{\mathbf{Q_C}} u(q)$), we have

Corollary 4 *Suppose that* $\exists x^*$ *such that* $F_u(x^*) = \varepsilon$. *Then*

$$\Pr\{\frac{vol_w\{u(q) \ge \hat{u}_1,\ q \in \mathbf{Q_C}\}}{vol_w\{\mathbf{Q_C}\}} \ge 1 - \varepsilon\} \ge 1 - \delta$$

if and only if $N_c \ge \frac{\ln \frac{1}{\delta}}{\ln \frac{1}{1-\varepsilon}}$.

To estimate the range of an uncertain quantity with a certain accuracy and confidence level apriori specified, we have the following corollary.

Corollary 5 *Suppose that $u(q) \neq constant$ in any open set of $\mathbf{Q_C}$. Then*

$$\Pr\{\frac{vol_w\{\hat{u}_1 < u(q) \leq \hat{u}_{N_c},\ q \in \mathbf{Q_C}\}}{vol_w\{\mathbf{Q_C}\}} \geq 1-\varepsilon\} \geq 1-\delta$$

if and only if $\mu(N_c) := (1-\varepsilon)^{N_c-1}\,[1+(N_c-1)\varepsilon] \leq \delta$.

Now we investigate the computational effort for the direct approach.

5.2 Global Sample Size

To estimate $\min_{\mathbf{Q_C}} u(q)$ (or determine parameter q achieving $\min_{\mathbf{Q_C}} u(q)$), we have

Theorem 3 *Suppose that $\exists x^*$ such that $F_u(x^*) = \varepsilon$. Then*

$$\Pr\left\{\frac{vol_w\{u(q) \geq \hat{u}_{min},\ q \in \mathbf{Q_C}\}}{vol_w\{\mathbf{Q_C}\}} \geq 1-\varepsilon\right\} \geq 1-\delta$$

if and only if $N \geq \dfrac{\ln(\frac{1}{\delta})}{\ln(\frac{1}{1-\rho\varepsilon})}$.

Proof.

$$\begin{aligned}
& \Pr\left\{\frac{vol_w\{u(q) \geq \hat{u}_{min},\ q \in \mathbf{Q_C}\}}{vol_w\{\mathbf{Q_C}\}} \geq 1-\varepsilon\right\} \\
= & \sum_{i=0}^{N} \Pr\{M=i\} \Pr\left\{\frac{vol_w\{u(q) \geq \hat{u}_{min},\ q \in \mathbf{Q_C}\}}{vol_w\{\mathbf{Q_C}\}} \geq 1-\varepsilon \mid M=i\right\} \\
= & \sum_{i=0}^{N} \binom{N}{i} \rho^i (1-\rho)^{N-i} \Pr\left\{\frac{vol_w\{u(q) \geq \hat{u}_{min},\ q \in \mathbf{Q_C}\}}{vol_w\{\mathbf{Q_C}\}} \geq 1-\varepsilon \mid M=i\right\}.
\end{aligned}$$

Notice that $\left\{\frac{vol_w\{u(q) \geq \hat{u}_{min},\ q \in \mathbf{Q_C}\}}{vol_w\{\mathbf{Q_C}\}} \geq 1-\varepsilon \mid M=i\right\} \iff \left\{\frac{vol_w\{u(q) \geq \hat{u}_1,\ q \in \mathbf{Q_C}\}}{vol_w\{\mathbf{Q_C}\}} \geq 1-\varepsilon \mid L \leq N\right\}$ with $N_c = i$ in the context of the indirect approach. By Corollary 1, we know that

$$\Pr\left\{\frac{vol_w\{u(q) \geq \hat{u}_1,\ q \in \mathbf{Q_C}\}}{vol_w\{\mathbf{Q_C}\}} \geq 1-\varepsilon \mid L \leq N\right\} = \Pr\left\{\frac{vol_w\{u(q) \geq \hat{u}_1,\ q \in \mathbf{Q_C}\}}{vol_w\{\mathbf{Q_C}\}} \geq 1-\varepsilon\right\}.$$

Apply Theorem 2 to the case of $N_c = i,\ m = 1$, we have

$$\Pr\left\{\frac{vol_w\{u(q) \geq \hat{u}_1,\ q \in \mathbf{Q_C}\}}{vol_w\{\mathbf{Q_C}\}} \geq 1-\varepsilon\right\} \geq 1-\mathcal{V}(i,1,\varepsilon) = 1-(1-\varepsilon)^i$$

with the equality holds if and only if $\exists x^*$ such that $F_u(x^*) = \varepsilon$. Therefore

$$\Pr\left\{\frac{vol_w\{u(q) \geq \hat{u}_{min},\ q \in \mathbf{Q_C}\}}{vol_w\{\mathbf{Q_C}\}} \geq 1-\varepsilon\right\} \geq \sum_{i=0}^{N} \binom{N}{i} \rho^i (1-\rho)^{N-i}[1-(1-\varepsilon)^i] = 1-(1-\varepsilon\rho)^N$$

with the equality holds if and only if $\exists x^*$ such that $F_u(x^*) = \varepsilon$. Finally, notice that $(1-\varepsilon\rho)^N \leq \delta$ if and only if $N \geq \frac{\ln(\frac{1}{\delta})}{\ln(\frac{1}{1-\rho\varepsilon})}$. This completes the proof. $\square$

It should be noted that sufficiency part of the preceding theorem has been obtained in [9] in the context of estimating robust stability margin. By the similar argument as that of Theorem 3, we have the following result for estimating $\max_{\mathbf{Q_C}} u(q)$ (or determine parameter q achieving $\max_{\mathbf{Q_C}} u(q)$).

Theorem 4 *Suppose that $\exists x^*$ such that $\Pr\{u(q) < x^* \mid q \in \mathbf{Q_C}\} = 1-\varepsilon$. Then*

$$\Pr\{\frac{vol_w\{u(q) \leq \hat{u}_{max},\ q \in \mathbf{Q_C}\}}{vol_w\{\mathbf{Q_C}\}} \geq 1-\varepsilon\} \geq 1-\delta$$

if and only if $N \geq \dfrac{\ln(\frac{1}{\delta})}{\ln(\frac{1}{1-\rho\varepsilon})}$.

To estimate the range of a quantity for the system under a certain constraint $\mathbf{C}$, we have

Theorem 5 *Suppose $u(q) \neq constant$ in any open set of $\mathbf{Q_C}$. Then*

$$\Pr\{\frac{vol_w\{\ \hat{u}_{min} < u(q) \leq \hat{u}_{max},\ q \in \mathbf{Q_C}\}}{vol_w\{\mathbf{Q_C}\}} \geq 1-\varepsilon\} = 1-\mu(N) \geq 1-\delta$$

if and only if $\mu(N) := (1-\varepsilon\rho)^{N-1}[1+(N-1)\varepsilon\rho] \leq \delta$.

Proof.

$$\begin{aligned}
& \Pr\left\{\frac{vol_w\{\hat{u}_{min} < u(q) \leq \hat{u}_{max},\ q \in \mathbf{Q_C}\}}{vol_w\{\mathbf{Q_C}\}} \geq 1-\varepsilon\right\} \\
= & \sum_{i=0}^{N} \Pr\{M=i\} \Pr\left\{\frac{vol_w\{\hat{u}_{min} < u(q) \leq \hat{u}_{max},\ q \in \mathbf{Q_C}\}}{vol_w\{\mathbf{Q_C}\}} \geq 1-\varepsilon \mid M=i\right\} \\
= & \sum_{i=0}^{N} \binom{N}{i} \rho^i (1-\rho)^{N-i} \Pr\left\{\frac{vol_w\{\hat{u}_{min} < u(q) \leq \hat{u}_{max},\ q \in \mathbf{Q_C}\}}{vol_w\{\mathbf{Q_C}\}} \geq 1-\varepsilon \mid M=i\right\}.
\end{aligned}$$

Notice that event $\left\{\frac{vol_w\{\hat{u}_{min}<u(q)\leq\hat{u}_{max},\ q\in\mathbf{Q_C}\}}{vol_w\{\mathbf{Q_C}\}} \geq 1-\varepsilon \mid M=i\right\}$ is equivalent to event

$$\left\{\frac{vol_w\{\hat{u}_1 < u(q) \leq \hat{u}_i,\ q \in \mathbf{Q_C}\}}{vol_w\{\mathbf{Q_C}\}} \geq 1-\varepsilon \mid L \leq N\right\}$$

with $N_c = i$ in the context of the indirect approach.

By Corollary 1 and Corollary 5, we have

$$\begin{aligned}
& \Pr\left\{\frac{vol_w\{\hat{u}_1 < u(q) \leq \hat{u}_i,\ q \in \mathbf{Q_C}\}}{vol_w\{\mathbf{Q_C}\}} \geq 1-\varepsilon \mid L \leq N\right\} \\
= & \Pr\left\{\frac{vol_w\{\hat{u}_1 < u(q) \leq \hat{u}_i,\ q \in \mathbf{Q_C}\}}{vol_w\{\mathbf{Q_C}\}} \geq 1-\varepsilon\right\} \\
= & 1-(1-\varepsilon)^{i-1}[1+(i-1)\varepsilon].
\end{aligned}$$

Therefore,

$$
\begin{aligned}
& \Pr\left\{ \frac{vol_w \left\{ \hat{u}_{min} < u(q) \leq \hat{u}_{max},\ q \in \mathbf{Q_C} \right\}}{vol_w \left\{ \mathbf{Q_C} \right\}} \geq 1 - \varepsilon \right\} \\
= & \sum_{i=0}^{N} \binom{N}{i} \rho^i (1-\rho)^{N-i} \left(1 - (1-\varepsilon)^{i-1} \left[1 + (i-1)\varepsilon \right] \right) \\
= & 1 - \sum_{i=0}^{N} \binom{N}{i} \rho^i (1-\rho)^{N-i} (1-\varepsilon)^{i-1} [1 + (i-1)\varepsilon] \\
= & 1 - \frac{1}{1-\varepsilon} \sum_{i=0}^{N} \binom{N}{i} ((1-\varepsilon)\rho)^i (1-\rho)^{N-i} + \frac{\varepsilon}{1-\varepsilon} \sum_{i=0}^{N} \binom{N}{i} ((1-\varepsilon)\rho)^i (1-\rho)^{N-i} \\
& - N\varepsilon\rho \sum_{i=1}^{N} \binom{N-1}{i-1} ((1-\varepsilon)\rho)^{i-1} (1-\rho)^{N-1-(i-1)} \\
= & 1 - \frac{1}{1-\varepsilon} (1-\varepsilon\rho)^N + \frac{\varepsilon}{1-\varepsilon} (1-\varepsilon\rho)^N - N\rho\varepsilon(1-\varepsilon\rho)^{N-1} \\
= & 1 - (1-\varepsilon\rho)^{N-1} [1 + (N-1)\varepsilon\rho] \\
= & 1 - \mu(N),
\end{aligned}
$$

which implies that

$$
\Pr\left\{ \frac{vol_w \left\{ \hat{u}_{min} < u(q) \leq \hat{u}_{max},\ q \in \mathbf{Q_C} \right\}}{vol_w \left\{ \mathbf{Q_C} \right\}} \geq 1 - \varepsilon \right\} \geq 1 - \delta
$$

if and only if $\mu(N) \leq \delta$. □

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